

Modeling and Optimal Control of Networks of Pipes and Canals

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Abstract

This paper deals with the optimal control of systems governed by non-linear systems of conservation laws at junctions. The applications considered range from gas compressors in pipelines to open channels management. The existence of an optimal control is proved. From the analytical point of view, these results are based on the well posedness of a suitable initial boundary value problem and on techniques for quasidifferential equations in a metric space.

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1 Introduction

The recent literature offers several results on the modeling of systems governed by conservation laws on networks. For instance, in [5, 6, 12, 13] the modeling of a network of gas pipelines is considered. The basic model is the p -system or, in [21], the full set of Euler equations. The key problem in these papers is the description of the evolution of fluid at a junction between two or more pipes. A different physical problem, leading to a similar analytical framework, is that of the flow of water in open channels, considered for example in [33].

Recent papers deal with the control of smooth solutions, see for instance [23, 27, 28, 29, 30, 33, 35, 36, 34]. Other approaches are based on suitable discretizations, as in [25, 39, 41, 44]. The present work presents a

general framework comprising several models in the existing literature and providing a proof of the existence of optimal controls for physically reasonable cost functions. In particular, in the structure below, the solution considered may well be *non smooth* and optimality is achieved in the set of all \mathbf{L}^1 controls with bounded variation.

As samples of the applications of the present results, we extend results from the current literature. First, we consider the optimal management of a compressor in a gas network. This device is required to compress fluid guaranteeing a given pressure while consuming the minimal energy, see [39, 44]. Then, the present framework is used to cover different optimization problems for the flow in open canals: the keeping of a constant water level through the optimal management of an underflow gate, see [23], and the prevention of overflow in a multiple valves system, see [42], or a pumping station, see [27].

From the analytical point of view, the above models are described by a system of conservation laws of the form

$$\begin{aligned} \partial_t u_l + \partial_x f_l(u_l) = g_l(t, x, u_l) \quad \text{with} \quad \begin{aligned} t &\in [0, +\infty[\\ x &\in [0, +\infty[\\ l &= 1, \dots, n. \end{aligned} \end{aligned} \quad (1.1)$$

Here, u_l is the vector of the conserved variables along the l -th pipe, f_l is a general nonlinear flux function and g_l is the source term related to the l -th tube. A time dependent interaction at the junction is described by time dependent conditions on the traces of the unknown variables at the junction, namely

$$\Psi(u_1(t, 0+), u_2(t, 0+), \dots, u_n(t, 0+)) = \Pi(t) \quad (1.2)$$

for a suitable smooth Ψ , see [5, 6, 12, 13, 21, 32]. In the examples below, $\Pi(t)$ is the control to be chosen in order to minimize a given cost functional $\mathcal{J} = \mathcal{J}(u, \Pi)$.

To obtain the existence of a control minimizing \mathcal{J} , we need first to prove the well posedness of (1.1)–(1.2). This is the content of our main analytical result, namely Theorem 2.3. The existence of an optimal control then follows in Proposition 2.4.

The next section is devoted to the analytical results. Section 3 presents the applications while all the technical details are collected in Section 4.

2 The Cauchy Problem at an Intersection

Throughout, we refer to [7] for the general theory of hyperbolic systems of conservation laws. Let $\Omega_l \subseteq \mathbb{R}^2$ be a non empty open set. Fix flows f_l such that $f \equiv (f_1, \dots, f_n)$ satisfies the following assumption at an n -tuple of states $\bar{u} \equiv (\bar{u}_1, \dots, \bar{u}_n) \in \Omega$, where $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_n$:

- (F) For $l = 1, \dots, n$, the flow f_l is in $\mathbf{C}^4(\Omega_l; \mathbb{R}^2)$, $Df_l(\bar{u}_l)$ admits a strictly negative eigenvector $\lambda_1^l(\bar{u}_l)$, a strictly positive one $\lambda_2^l(\bar{u}_l)$ and each characteristic field is either genuinely nonlinear or linearly degenerate.

Under this condition, when $g_l = 0$ and $x \in \mathbb{R}$, (1.1) generates a Standard Riemann Semigroup, see [7, Chapter 8]. Recall that a 2×2 system of conservation laws admits entropies, see [43, Paragraph 9.3].

Here and in what follows, $\mathbb{R}^+ = [0, +\infty[$. For later use, with a slight abuse of notation, we denote by

$$\begin{aligned} \|u\| &= \sum_{l=1}^n \|u_l\| & \text{for } u &\in \Omega \\ \|u\|_{\mathbf{L}^1} &= \int_{\mathbb{R}^+} \|u(x)\| dx & \text{for } u &\in \mathbf{L}^1(\mathbb{R}^+; \Omega), \\ \text{TV}(u) &= \sum_{l=1}^n \text{TV}(u_l) & \text{for } u &\in \mathbf{BV}(\mathbb{R}^+; \Omega). \end{aligned}$$

Below, the constant state $\bar{u} \in \Omega$, is fixed. Throughout, we also fix a time $\hat{T} \in]0, +\infty]$ and a positive $\hat{\delta}$. For all $\delta \in]0, \hat{\delta}]$, we denote

$$\mathcal{U}_\delta = \left\{ u \in \bar{u} + \mathbf{L}^1(\mathbb{R}^+; \Omega) : \text{TV}(u) \leq \delta \right\}.$$

On the source term $g \equiv (g_1, \dots, g_n)$ we require that if G is defined by $(G(t, u))(x) = (g_1(t, x, u_1(x)), \dots, g_n(t, x, u_n(x)))$, then G satisfies:

- (G) $G: [0, \hat{T}] \times \mathcal{U}_{\hat{\delta}} \mapsto \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^{2n})$ is such that there exist positive L_1, L_2 and for all $t \in [0, \hat{T}]$

$$\begin{aligned} \forall u, w \in \mathcal{U}_{\hat{\delta}} \quad \|G(t, u) - G(s, w)\|_{\mathbf{L}^1} &\leq L_1 \cdot (\|u - w\|_{\mathbf{L}^1} + |t - s|) \\ \forall u \in \mathcal{U}_{\hat{\delta}} \quad \text{TV}(G(t, u)) &\leq L_2. \end{aligned}$$

Below, we require only (G), thus comprising also non-local terms, see [18]. Examples of sources g such that the corresponding G satisfies (G) are provided by the next proposition, which comprehends the applications below.

Proposition 2.1 *Assume that the map $g: \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}^{2n}$ satisfies:*

1. *there exists a state \bar{u} and a compact subset $\bar{\mathcal{K}}$ of \mathbb{R}^+ such that $g(x, \bar{u}) = 0$ for all $x \in \mathbb{R}^+ \setminus \bar{\mathcal{K}}$;*
2. *there exists a finite positive measure μ such that for all $x_1, x_2 \in \mathbb{R}^+$ with $x_1 \leq x_2$, and all $u \in \Omega^l$,*

$$\|g_l(x_2+, u) - g_l(x_1-, u)\| \leq \mu([x_1, x_2]) ;$$

3. *there exists a positive \hat{L} such that for all $u, w \in \Omega$, for all $x \in \mathbb{R}^+$,*

$$\|g(x, u) - g(x, w)\| \leq \hat{L} \cdot \|u - w\|.$$

Then, condition **(G)** is satisfied.

The proof is deferred to Section 4.

We consider the Cauchy problem at a junction, see [11, 13, 20, 22, 31]. First, we extend [20, Definition 3.1] to the present case of a Cauchy problem with sources.

Definition 2.2 Fix the maps $\Psi \in \mathbf{C}^1(\Omega; \mathbb{R}^n)$ and $\Pi \in \mathbf{BV}(\mathbb{R}^+; \mathbb{R}^n)$. A weak solution on $[0, T]$ to

$$\begin{cases} \partial_t u_l + \partial_x f_l(u_l) = g_l(t, x, u_l) & t \in \mathbb{R}^+ \quad l \in \{1, \dots, n\} \\ \Psi(u(t, 0)) = \Pi(t) & x \in \mathbb{R}^+ \quad u_o \in \bar{u} + \mathbf{L}^1(\mathbb{R}^+; \Omega) \\ u(0, x) = u_o(x) \end{cases} \quad (2.1)$$

is a map $u \in \mathbf{C}^0([0, T]; \bar{u} + \mathbf{L}^1(\mathbb{R}^+; \Omega))$ such that for all $t \in [0, T]$, $u(t) \in \mathbf{BV}(\mathbb{R}^+; \Omega)$ and

(W) $u(0) = u_o$ and for all $\varphi \in \mathbf{C}_c^\infty(]0, T[\times]0, +\infty[; \mathbb{R})$ and for $l = 1, \dots, n$

$$\int_0^T \int_{\mathbb{R}^+} (u_l \partial_t \varphi + f_l(u_l) \partial_x \varphi) dx dt + \int_0^T \int_{\mathbb{R}^+} \varphi(t, x) g_l(t, x, u_l) dx dt = 0.$$

(Ψ) The condition at the junction is met: for a.e. $t \in \mathbb{R}^+$, $\Psi(u(t, 0+)) = \Pi(t)$.

The weak solution (ρ, q) is an entropy solution if for any entropy – entropy flux pair (η_l, q_l) , for all $\varphi \in \mathbf{C}_c^\infty(]0, T[\times]0, +\infty[; \mathbb{R}^+)$ and for $l = 1, \dots, n$

$$\int_0^T \int_{\mathbb{R}^+} (\eta_l(u_l) \partial_t \varphi + q_l(u_l) \partial_x \varphi) dx dt + \int_0^T \int_{\mathbb{R}^+} D\eta_l(u_l) g(t, x, u) \varphi dx dt \geq 0.$$

We are now ready to state the main result of this paper, namely the well posedness of the Cauchy Problem for (2.1) at the junction.

Below, we denote by $r_2^l(u)$ the right eigenvector of $Df_l(u)$ corresponding to the second characteristic family.

As is usual in the context of initial boundary value problems, [1, 2, 3], we consider the metric space $X = (\bar{u} + \mathbf{L}^1(\mathbb{R}^+, \Omega)) \times (\bar{\Pi} + \mathbf{L}^1(\mathbb{R}^+, \mathbb{R}^n))$ equipped with the \mathbf{L}^1 distance. Let the extended variable $\mathbf{p} \equiv (u, \Pi)$ with $u = u(x)$, respectively $\Pi = \Pi(t)$, be defined for $x \geq 0$, respectively $t \geq 0$. Correspondingly, denote

$$\begin{aligned} d_X((u, \Pi), (\tilde{u}, \tilde{\Pi})) &= \|(u, \Pi) - (\tilde{u}, \tilde{\Pi})\|_X = \|u - \tilde{u}\|_{\mathbf{L}^1} + \|\Pi - \tilde{\Pi}\|_{\mathbf{L}^1} \\ \text{TV}(\mathbf{p}) &= \text{TV}(u) + \text{TV}(\Pi) + \|\Psi(u(0+)) - \Pi(0+)\| \\ \mathcal{D}^\delta &= \{\mathbf{p} \in X : \text{TV}(\mathbf{p}) \leq \delta\} \end{aligned} \quad (2.2)$$

Below, \mathcal{T}_t is the right translation, i.e. $(\mathcal{T}_t \Pi)(s) = \Pi(t + s)$.

Theorem 2.3 *Let $n \in \mathbb{N}$, $n \geq 2$ and assume that f satisfies **(F)** at \bar{u} and G satisfies **(G)**. Fix a map $\Psi \in \mathbf{C}^1(\Omega; \mathbb{R}^n)$ that satisfies*

$$\det \begin{bmatrix} D_1 \Psi(\bar{u}) r_2^1(\bar{u}_1) & D_2 \Psi(\bar{u}) r_2^2(\bar{u}_2) & \dots & D_n \Psi(\bar{u}) r_2^n(\bar{u}_n) \end{bmatrix} \neq 0 \quad (2.3)$$

where $D_l \Psi = D_{u_l} \Psi$, and let $\bar{\Pi} = \Psi(\bar{u})$. Then, there exist positive δ, δ', L, T , domains \mathcal{D}_t , for $t \in [0, T]$, and a map

$$\mathcal{E}: \{(\tau, t_o, \mathbf{p}): t_o \in [0, T[, \tau \in [0, T - t_o], \mathbf{p} \in \mathcal{D}_{t_o}\} \mapsto \mathcal{D}^\delta$$

such that:

1. $\mathcal{D}^{\delta'} \subseteq \mathcal{D}_t \subseteq \mathcal{D}^\delta$ for all $t \in [0, T]$;
2. for all $t_o \in [0, T]$ and $\mathbf{p} \in \mathcal{D}_{t_o}$, $\mathcal{E}(0, t_o)\mathbf{p} = \mathbf{p}$;
3. for all $t_o \in [0, T]$ and $\tau \in [0, T - t_o]$, $\mathcal{E}(\tau, t_o)\mathcal{D}_{t_o} \subseteq \mathcal{D}_{t_o+\tau}$;
4. for all $t_o \in [0, T]$, $\tau_1, \tau_2 \geq 0$ with $\tau_1 + \tau_2 \in [0, T - t_o]$,

$$\mathcal{E}(\tau_2, t_o + \tau_1) \circ \mathcal{E}(\tau_1, t_o) = \mathcal{E}(\tau_2 + \tau_1, t_o);$$
5. for all $(u_o, \Pi) \in \mathcal{D}_{t_o}$, set $\mathcal{E}(t, t_o)(u_o, \Pi) = (u(t), \mathcal{T}_t \Pi)$ where $t \mapsto u(t)$ is the entropy solution to the Cauchy Problem (2.1) according to Definition 2.2 while the second component $t \mapsto \mathcal{T}_t \Pi$ is the right translation;
6. \mathcal{E} is tangent to Euler polygonal, in the sense that for all $t_o \in [0, T]$, for all $(u_o, \Pi) \in \mathcal{D}_{t_o}$, setting $\mathcal{E}(t, t_o)(u_o, \Pi) = (u(t), \mathcal{T}_t \Pi)$,

$$\lim_{t \rightarrow 0} \frac{1}{t} \left\| u(t) - (S_t(u_o, \Pi) + t G(t_o, u_o)) \right\|_{\mathbf{L}^1} = 0$$

where S is the semigroup generated by the convective part in (2.1);

7. for all $t_o \in [0, T]$, $\tau \in [0, T - t_o]$ and for all $\mathbf{p}, \tilde{\mathbf{p}} \in \mathcal{D}_{t_o}$,

$$\begin{aligned} \left\| \mathcal{E}(\tau, t_o)\mathbf{p} - \mathcal{E}(\tau, t_o)\tilde{\mathbf{p}} \right\|_{\mathbf{L}^1} &\leq L \cdot \|u - \tilde{u}\|_{\mathbf{L}^1} \\ &+ L \cdot \int_{t_o}^{t_o+\tau} \left\| \tilde{\Pi}(t) - \Pi(t) \right\| dt. \end{aligned} \quad (2.4)$$

The proof is deferred to Section 4. Note that in the case $n = 2$, $f_1 = -f_2$, $\Pi = 0$ and $\Psi(u_1, u_2) = f_1(u_1) + f_2(u_2)$ we (re)obtain the well posedness of a standard 2×2 balance law.

From 7. in Theorem 2.3, we immediately obtain the following existence result for an optimal control function Π to the nonlinear constrained optimization problem

$$\text{minimize } \mathcal{J}(\Pi) \text{ subject to } \begin{cases} \partial_t u_l + \partial_x f_l(u_l) = g_l(t, x, u_l) \\ \Psi(u(t, 0)) = \Pi(t) \\ u(0, x) = u_o(x) \end{cases} \quad \text{on } [t_o, T].$$

Proposition 2.4 *Let $n \in \mathbb{N}$, $n \geq 2$. Assume that f satisfies **(F)** at \bar{u} and G satisfies **(G)**. Fix a map $\Psi \in \mathbf{C}^1(\Omega; \mathbb{R}^n)$ satisfying (2.3) and let $\bar{\Pi} = \Psi(\bar{u})$. With the notation in Theorem 2.3, for a fixed $u_o \in \mathcal{U}_\delta$, assume that*

$$\begin{aligned} J_o & : \left\{ \Pi_{|[0,T]} : \Pi \in \left(\bar{\Pi} + \mathbf{L}^1(\mathbb{R}^+; \mathbb{R}^n) \right) \text{ and } (u_o, \Pi) \in \mathcal{D}^\delta \right\} \mapsto \mathbb{R} \\ J_1 & : \mathcal{D}^\delta \mapsto \mathbb{R} \end{aligned}$$

are non negative and lower semicontinuous with respect to the \mathbf{L}^1 norm. Then, the cost functional

$$\mathcal{J}(\Pi) = J_o(\Pi) + \int_0^T J_1(\mathcal{E}(\tau, 0)(u_o, \Pi)) d\tau \quad (2.5)$$

admits a minimum on $\left\{ \Pi \in (\bar{\Pi} + \mathbf{L}^1([0, T]; \mathbb{R}^n)) : (u_o, \Pi) \in \mathcal{D}_0 \right\}$.

Proof. Due to Theorem 2.3, $\mathcal{E}(\cdot, 0)u_o$ is Lipschitz continuous in Π . An application of Fatou's Lemma shows that the second summand in (2.5) is lower semicontinuous. Hence, also \mathcal{J} is lower semicontinuous and the existence of a minimizer follows from Weierstraß Theorem. \square

3 Networks of Gas Pipelines and Open Canals

3.1 Compressor Control for Gas Networks

We describe a compressor acting between two pipes with the same cross section by the equations

$$\begin{cases} \partial_t \rho_l + \partial_x q_l = 0, \\ \partial_t q_l + \partial_x \left(\frac{q_l^2}{\rho_l} + p(\rho_l) \right) = -\chi_{[0, \mathbf{L}]}(x) \nu \frac{q_l |q_l|}{\rho_l} - g \rho_l \sin \alpha_l(x) \end{cases} \quad (3.1)$$

$$t \in \mathbb{R}^+, \quad x \in \mathbb{R}^+, \quad l = 1, 2, \quad (\rho_l, q_l) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}.$$

where ρ is the mass density of a given fluid, q its linear momentum density, ν accounts for the friction against the pipe's walls, g is gravity and $\alpha(x)$ is the inclination of the pipe at x . It is reasonable to assume that $\alpha(x) = 0$ for x sufficiently large. Furthermore, we are interested only in the dynamics in the pipe up to the maximum length \mathbf{L} . The pressure law $p = p(\rho)$ satisfies

(P) $p \in \mathbf{C}^2(\mathbb{R}^+; \mathbb{R}^+)$, $p(0) = 0$ and for all $\rho \in \mathbb{R}^+$, $p'(\rho) > 0$, $p''(\rho) \geq 0$.

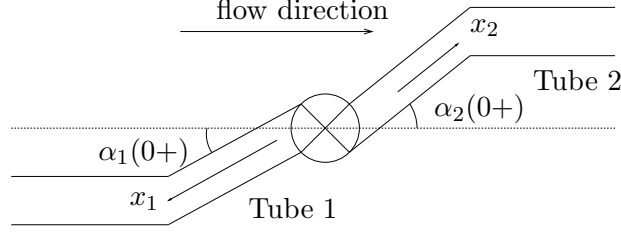


Figure 1: Notation for the compressor model (3.1).

As usual in the engineering literature, we focus below on the γ law

$$p(\rho) = p_* \cdot \left(\frac{p}{p_*} \right)^\gamma \quad (3.2)$$

for suitable positive constants p_*, ρ_* . Choosing an initial datum $\bar{u}_l = (\bar{\rho}_l, \bar{q}_l)$ in the *subsonic* region

$$\Omega^l = \left\{ (\rho, q) \in \mathring{\mathbb{R}}^+ \times \mathbb{R} : \lambda_1(\rho, q) < 0 < \lambda_2(\rho, q) \right\},$$

ensures that **(F)** holds at \bar{u} . Recall the standard relations

$$\begin{aligned} \lambda_1(\rho, q) &= (q/\rho) - \sqrt{p'(\rho)}, & \lambda_2(\rho, q) &= (q/\rho) + \sqrt{p'(\rho)}. \\ r_1(\rho, q) &= \begin{bmatrix} -1 \\ -\lambda_1(\rho, q) \end{bmatrix} & r_2(\rho, q) &= \begin{bmatrix} 1 \\ \lambda_2(\rho, q) \end{bmatrix} \end{aligned} \quad (3.3)$$

The coupling condition describe the effect of a compressor sited at the junction between the 2 pipes. A standard relation in the engineering literature is the following, see also [40, Section 4.4, Formula (4.9)] or [44]:

$$\Psi(u_1, u_2) = \begin{bmatrix} q_1 + q_2 \\ q_2 \left(\left(\frac{p(\rho_2)}{p(\rho_1)} \right)^{(\gamma-1)/\gamma} - 1 \right) \end{bmatrix} \text{ and } \Pi(t) = \begin{bmatrix} 0 \\ \Pi_2(t) \end{bmatrix} \quad (3.4)$$

and Π_2 is proportional to the applied compressor power.

Proposition 3.1 *Let $\bar{u}_1, \bar{u}_2 \in \Omega$ satisfy*

$$\Psi(\bar{u}_1, \bar{u}_2) = \begin{bmatrix} 0 \\ \bar{\Pi}_2 \end{bmatrix}$$

for a positive $\bar{\Pi}_2$. Then, Theorem 2.3 applies to (3.1)–(3.2)–(3.4).

Proof. By (3.3), **(F)** is satisfied at \bar{u} . Condition **(G)** is satisfied due to Proposition 2.1. Finally, condition (2.3) leads to the determinant

$$\begin{aligned} & \det \left[\begin{bmatrix} 0 & 1 \\ \partial_{\rho_1} \Psi_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda_2(\bar{u}_1) \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ \partial_{\rho_2} \Psi_2 & \partial_{q_2} \Psi_2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda_2(\bar{u}_2) \end{bmatrix} \right] \\ &= \lambda_2(\bar{u}_1) (\lambda_2(\bar{u}_2) \partial_{q_2} \Psi_2 + \partial_{\rho_2} \Psi_2) - \lambda_2(\bar{u}_2) \partial_{\rho_1} \Psi_2 > 0 \end{aligned}$$

due to the choice $\bar{u}_1, \bar{u}_2 \in \Omega$. Hence, Theorem 2.3 applies. \square

A typical optimization problem in gas networks [39, 44] is the control of compressors stations such that a certain outlet pressure \bar{p} for a customer located in the interval $[x_a, x_b], x_a > 0$, is satisfied. In the optimization problem we penalize large energy consumption by the \mathbf{L}^∞ -norm, frequent changes in the applied compressor energy by the TV-norm and deviations from the desired outlet pressure. We model this situation by considering

$$\begin{aligned} J_o(\Pi) &= \text{TV}(\Pi) + \|\Pi\|_{\mathbf{L}^\infty} \quad \text{and} \\ J_1(\mathcal{E}(\tau, 0)(u_o, \Pi)) &= \int_{x_a}^{x_b} |p(\rho_2(\tau, x)) - \bar{p}| dx, \end{aligned}$$

where $\rho_2(t, x)$ is given by the solution of (3.1). The lower semicontinuity of J_o is obvious. J_1 is \mathbf{L}^1 -Lipschitz. Indeed, let ρ_2 and $\tilde{\rho}_2$ denote the density distribution in the pipe $l = 2$ corresponding to the same initial datum and to the controls Π and $\tilde{\Pi}$. Then,

$$\begin{aligned} & \left| J_1(\mathcal{E}(\tau, 0)(u_o, \Pi)) - J_1(\mathcal{E}(\tau, 0)(u_o, \tilde{\Pi})) \right| \\ & \leq \int_0^T \int_{x_a}^{x_b} \left| |p(\rho_2(t_0 + \tau, x)) - \bar{p}| - |p(\tilde{\rho}_2(t_0 + \tau, x)) - \bar{p}| \right| dx d\tau \\ & \leq C \int_0^T \int_{x_a}^{x_b} |\rho_2(\tau, x) - \tilde{\rho}_2(\tau, x)| dx dt \\ & \leq C \left\| \Pi_2 - \tilde{\Pi}_2 \right\|_{\mathbf{L}^1([0, T])}, \end{aligned}$$

for some constant C and due to (2.4). Hence, Proposition 2.4 applies.

3.2 Control of Open Canals

Similarly to the models in [23, 27, 33], we consider canals with fixed rectangular cross section having width b_l described by

$$\begin{cases} \partial_t H_l + \partial_x Q_l = 0 \\ \partial_t Q_l + \partial_x \left(\frac{Q_l^2}{H_l} + \frac{g}{2} H_l^2 \right) = -g H_l \sin \alpha_l(x) - \chi_{[0, \mathbf{L}]}(x) \nu \frac{Q_l |Q_l|}{H_l} \end{cases} \quad (3.5)$$

where

$$t \in \mathbb{R}^+, \quad x \in \mathbb{R}^+, \quad l = 1, \dots, n, \quad (H_l, Q_l) \in \mathring{\mathbb{R}}^+ \times \mathbb{R}$$

and $H_l(t, x)$ is the level of water at time t , point x in canal l ; $b_l Q_l$ is the total water flow; α_l is the inclination, g gravity and \mathbf{L} is the length of the canal as in Section 3.1. Note that (3.1) reduces to (3.5) in the case $\gamma = 2$.

3.2.1 The Case of an Underflow Gate

Following [23], consider (3.5) for $n = 2$ with the coupling condition

$$\Psi(u_1, u_2) = \begin{bmatrix} b_1 Q_1 + b_2 Q_2 \\ \frac{Q_1^2}{H_1 - H_2} \end{bmatrix} \quad \text{and} \quad \Pi(t) = \begin{bmatrix} 0 \\ u(t) \end{bmatrix} \quad (3.6)$$

the control u being the opening of the underflow gate, see Figure 2. The

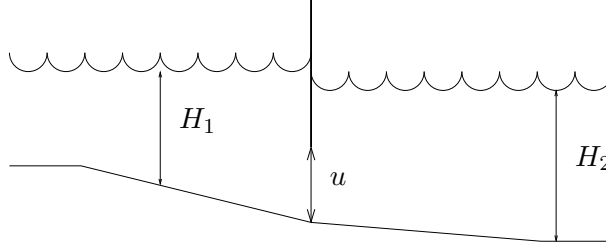


Figure 2: Notation for (3.5) with coupling conditions (3.6).

conditions **(F)** and **(G)** are proved as above. The determinant in (2.3) gives:

$$\begin{aligned} \det & \left[\begin{bmatrix} 0 & b_1 \\ -\frac{Q_1^2}{(H_1 - H_2)^2} & \frac{2Q_1}{H_1 - H_2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda_2(\bar{u}_1) \end{bmatrix} \begin{bmatrix} 0 & b_2 \\ \frac{Q_1^2}{(H_1 - H_2)^2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda_2(\bar{u}_2) \end{bmatrix} \right] \\ &= \frac{(b_1 \lambda_2(\bar{u}_1) + b_2 \lambda_2(\bar{u}_2)) Q_1^2}{(H_1 - H_2)^2} - \frac{2b_2 \lambda_2(\bar{u}_1) \lambda_2(\bar{u}_2) Q_1}{H_1 - H_2} > 0. \end{aligned}$$

The determinant is positive, since the underflow gates are only operating for $H_1 > H_2$ and (due to our parametrization of the pipe) for $Q_1 \leq 0$. Finally, Theorem 2.3 applies for sub-critical data \bar{u} .

A typical problem for horizontal pipes is to maintain a steady height in the downstream canal $l = 2$. Hence, we consider equation (3.5) with $\alpha_l \equiv 0$ and penalize large gradients in the water height. We introduce the cost functionals

$$J_o = \int_0^T |u(t)| dt \quad \text{and} \quad J_1 = \int_0^{+\infty} \varphi(x) d|\partial_x H_2|. \quad (3.7)$$

Here, H_2 is the water height in canal $l = 2$ given by the solution to (3.5) and (3.6). The non negative and lower semicontinuous weight φ assigns different importance to oscillations in the water level at different locations. Under the assumptions of Theorem 2.3, the map $x \rightarrow H_2(t, x)$ is a function of bounded variation and hence, $\partial_x H_2(t, x)$ is a Radon measure. Then, the measure $|\partial_x H_2|$ is the total variation of $\partial_x H_2$. Due to (2.3) we obtain that $\Pi_k \rightarrow \Pi^*$ in \mathbf{L}^1 implies that for any fixed $t \in [0, T]$ we have $H_2^k(t, \cdot) \rightarrow H_2^*(t, \cdot)$ in \mathbf{L}^1 . Therefore, the same arguments as in [14, Theorem 2.2] and [14, Lemma 2.1] show that Proposition 2.4 can be applied to (3.5)–(3.7).

3.2.2 The Case of Multiple Valves

We consider conditions for valve control similar to those introduced in [9, Section 2.9] or [10] and discuss a situation with n connected pipes as in Figure 3. We control the inflow at each connected pipe by the opening of a flow control valve [42]. This amounts to

$$\Psi(u_1, \dots, u_n) = \begin{bmatrix} \sum_{i=1}^n b_i Q_i \\ Q_1 \\ \vdots \\ Q_{n-1} \end{bmatrix} \quad \text{and} \quad \Pi(t) = \begin{bmatrix} 0 \\ u_1(t) \\ \vdots \\ u_{n-1}(t) \end{bmatrix}. \quad (3.8)$$

The assertions of Theorem 2.3 are satisfied, since the determinant in (2.3)

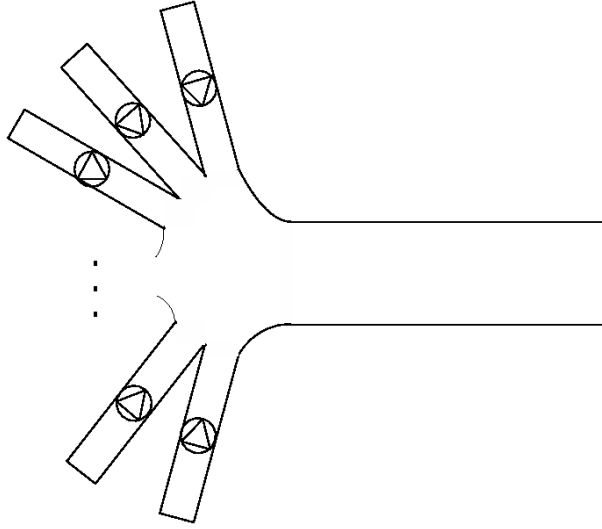


Figure 3: Illustration of a multiple valves junction. The outflow in the large canal is controlled through $n - 1$ valves at the incoming pipes.

evaluates to $\prod_{i=1}^n b_i \lambda_2(\bar{u}_i) \neq 0$ for any sub-critical state \bar{u} .

We consider the problem to prevent overflow in the downstream canal n by valve control at the node. We assume costs associated with the operation of valves given by non-negative, bounded functions $c_i(t)$ for $i = 1, \dots, n - 1$ and a maximal height of the water \bar{h} . We model this problem by minimizing

$$J_o = \sum_{i=1}^{n-1} \int_0^T c_i(t) u_i(t) dt \quad \text{and} \quad J_1 = \int_0^{\mathbf{L}} (H_n - \bar{h})^+ dx. \quad (3.9)$$

Herein, H_n is the solution to (3.5) and (3.8) and $x^+ = \max\{0, x\}$. Since $x \rightarrow x^+$ is Lipschitz, we can apply the same arguments as in Section 3.1 and obtain that Proposition 2.4 applies in this case.

3.2.3 The Case of a Pumping Station

We consider the case of a simple pumping station, i.e. we supply (3.5) with $n = 2$ and with the coupling conditions from [27]:

$$\Psi(u_1, u_2) = \begin{bmatrix} b_1 Q_1 + b_2 Q_2 \\ H_1 - H_2 \end{bmatrix} \quad \Pi(t) = \begin{bmatrix} 0 \\ u(t) \end{bmatrix}$$

In the present case, conditions **(F)** and **(G)** are proved as in Proposition 3.1. A direct computation allows to verify that the determinant in (2.3) is non-zero and therefore Theorem 2.3 applies. A reasonable optimization problem consists in minimizing \mathcal{J} for (3.9).

4 Technical Details

As a general reference on the theory of hyperbolic systems of conservation laws, we refer to [7]. As usual, C denotes a sufficiently large constant dependent only on f restricted to a neighborhood of the initial states.

Proof of Proposition 2.1. Note first that 1. and 3. imply that G attains values in \mathbf{L}^1 . Concerning the bound on the total variation, by 2.,

$$\begin{aligned} \text{TV}(G(u)) &= \sup \sum_i \left\| g(x_i, u(x_i)) - g(x_{i-1}, u(x_{i-1})) \right\| \\ &\leq \sup \sum_i \left\| g(x_i, u(x_i)) - g(x_{i-1}, u(x_i)) \right\| \\ &\quad + \sup \sum_i \left\| g(x_{i-1}, u(x_i)) - g(x_{i-1}, u(x_{i-1})) \right\| \\ &\leq \sup \sum_i \mu([x_{i-1}, x_i]) + \hat{L} \sup \sum_i \|u(x_i) - u(x_{i-1})\| \\ &\leq 2\mu(\mathbb{R}^+) + \hat{L} \text{TV}(u). \end{aligned}$$

Condition 4. directly implies the \mathbf{L}^1 -Lipschitz condition on G . \square

4.1 The Convective Part

This section is devoted to the Cauchy problem for (1.1)–(1.2) in the case $g_l \equiv 0$ for $l = 1, \dots, n$. First, we rewrite it as an initial – boundary value problem for a $(2n) \times (2n)$ system of hyperbolic conservation laws. To this aim, introduce positive

$$\lambda_{\min}^l < \min_{i=1,2} \inf_{u \in \Omega_l} \left| \lambda_i^l(u) \right| \quad \text{and} \quad \lambda_{\max}^l > \max_{i=1,2} \sup_{u \in \Omega_l} \left| \lambda_i^l(u) \right|$$

for $l = 1, \dots, n$. Introduce the flow

$$F_{2l-1}(U) = \Delta_l \cdot (f_l)_1(U_{2l-1}, U_{2l}), \quad F_{2l}(U) = \Delta_l \cdot (f_l)_2(U_{2l-1}, U_{2l}) \quad (4.1)$$

where the dilatation factor Δ_l are recursively defined by

$$\Delta_1 = \frac{1}{\lambda_{\min}^1} \quad \text{and} \quad \Delta_l = \frac{\lambda_{\max}^{l-1}}{\lambda_{\min}^l} \Delta_{l-1} \quad \text{for } l = 2, \dots, n$$

Lemma 4.1 *The flow (4.1) defines a hyperbolic $(2n) \times (2n)$ system of conservation laws, the eigenvalues $\Lambda_1, \dots, \Lambda_{2n}$ of DF satisfy, for $l = 1, \dots, n$,*

$$\begin{aligned} -\frac{\lambda_{\max}^1}{\lambda_{\min}^1} &\leq \Lambda_1 < -1 \\ 1 &< \Lambda_2 \leq \frac{\lambda_{\max}^1}{\lambda_{\min}^1} \\ -\prod_{k=1}^l \frac{\lambda_{\max}^k}{\lambda_{\min}^k} &\leq \Lambda_{2l-1} < -\prod_{k=1}^{l-1} \frac{\lambda_{\max}^k}{\lambda_{\min}^k} \\ \prod_{k=1}^{l-1} \frac{\lambda_{\max}^k}{\lambda_{\min}^k} &< \Lambda_{2l} \leq \prod_{k=1}^l \frac{\lambda_{\max}^k}{\lambda_{\min}^k} \end{aligned}$$

Proposition 4.2 *The Cauchy problem at the junction and the IBVP*

$$\left\{ \begin{array}{l} \partial_t u_l + \partial_x f_l(u_l) = 0 \\ \Psi(u(0+, t)) = \Pi(t) \\ u(0, x) = u_o(x) \end{array} \right. \quad \left\{ \begin{array}{l} \partial_t U + \partial_x F(U) = 0 \\ \Psi(U(0+, t)) = \Pi(t) \\ U(0, x) = U_o(x) \end{array} \right.$$

both defined for $t \geq 0$ and $x > 0$, with

$$(U_o)_{2l-1}(x) = (u_o)_{2l-1}(x/\Delta_l) \quad (U_o)_{2l}(x) = (u_o)_{2l}(x/\Delta_l)$$

are equivalent, in the sense that $u = u(t, x)$ solves the former problem in the sense of Definition 2.2 if and only if the map $U = U(t, x)$ defined by

$$U_{2l-1}(x) = u_{2l-1}(x/\Delta_l) \quad U_{2l}(x) = u_{2l}(x/\Delta_l)$$

is a weak entropy solution to the latter problem.

For the definition of weak entropy solution to the IBVP above, see [26] or [1, 3, 24]. The proof of Proposition 4.2 is immediate. Note that the two problems differ by a linear change of coordinates in the space variables, hence the entropicity of solutions is maintained.

4.1.1 The Riemann Problem at a Junction

Let f satisfy **(F)** at \bar{u} and let $\Psi \in \mathbf{C}^1(\Omega; \mathbb{R}^n)$. By *Riemann Problem at the Junction* we mean the problem

$$\begin{cases} \partial_t u_l + \partial_x f_l(u_l) = 0 \\ \Psi(u(t, 0+)) = \Pi \\ u_l(0, x) = u_{o,l}, \end{cases} \quad \begin{matrix} t \in \mathbb{R}^+ \\ x \in \mathbb{R}^+ \end{matrix} \quad \begin{matrix} l \in \{1, \dots, n\} \\ u_l \in \Omega^l \end{matrix} \quad (4.2)$$

where, for $l = 1, \dots, n$, $u_{o,l}$ are constant in Ω^l and $\Pi \in \mathbb{R}^n$ is also a constant.

Definition 4.3 Fix the map $\Psi \in \mathbf{C}^1(\Omega; \mathbb{R}^n)$. A solution to the Riemann Problem (4.2) is a function $u: \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \Omega$ such that

(L) For $l = 1, \dots, n$, the function $(t, x) \mapsto u_l(t, x)$ is self-similar and coincides with the restriction to $x > 0$ of the Lax solution to the standard Riemann Problem

$$\begin{cases} \partial_t u_l + \partial_x f_l(u_l) = 0 \\ u_l(0, x) = \begin{cases} u_{o,l} & \text{if } x > 0 \\ u_l(1, 0+) & \text{if } x < 0. \end{cases} \end{cases}$$

(Ψ) The trace $u(t, 0+)$ of u at the junction satisfies $\Psi(u(t, 0+)) = \Pi$ for all $t > 0$.

The following proposition yields the continuous dependence of the solution to the Riemann problem from the initial state, from the coupling condition Ψ and from the control term Π .

Proposition 4.4 Let $n \in \mathbb{N}$ with $n \geq 2$, and f satisfy **(F)** at \bar{u} . Fix $\Psi \in \mathbf{C}^1(\Omega^n; \mathbb{R}^n)$ satisfying (2.3) and a constant $\bar{\mathbf{p}} = (\bar{u}, \bar{\Pi})$ with $\bar{\Pi} = \Psi(\bar{u})$. Then, there exist positive δ, K such that

1. for all $\mathbf{p} \equiv (u_o, \Pi)$ with $\|\mathbf{p} - \bar{\mathbf{p}}\| < \delta$, the Riemann Problem (4.2) admits a unique self-similar solution $(t, x) \mapsto (\mathcal{R}(\mathbf{p}))(t, x)$ in the sense of Definition 4.2;
2. let $\mathbf{p}, \tilde{\mathbf{p}}$ satisfy $\|\mathbf{p} - \bar{\mathbf{p}}\| < \delta$ and $\|\tilde{\mathbf{p}} - \bar{\mathbf{p}}\| < \delta$. Then, the traces at the junction of the corresponding solutions to (4.2) satisfy

$$\left\| (\mathcal{R}(\mathbf{p}))(t, 0+) - (\mathcal{R}(\tilde{\mathbf{p}}))(t, 0+) \right\| \leq K \cdot \|\mathbf{p} - \tilde{\mathbf{p}}\|; \quad (4.3)$$

3. call $\Sigma(\mathbf{p})$ the n -vector of the total sizes of the 2-waves in the solution to (4.2). Then,

$$\|\Sigma(\mathbf{p}) - \Sigma(\tilde{\mathbf{p}})\| \leq K \cdot \|\mathbf{p} - \tilde{\mathbf{p}}\|.$$

The proof is omitted, since it follows from [1, Lemma 2.2] through Proposition 4.2 or from simple modifications of [20, Proposition 2.2].

4.1.2 The Cauchy Problem at a Junction

For a piecewise constant function $u = \sum_{\alpha} u^{\alpha} \chi_{[x^{\alpha-1}, x^{\alpha}]}$ the usual Glimm functionals in the case of a non characteristic boundary, see [2, Lemma 4] and [1, 24], take the form

$$\begin{aligned} V(u) &= \sum_{\alpha, l} \left(2 K_J \cdot \left| \sigma_{1, \alpha}^l \right| + \left| \sigma_{2, \alpha}^l \right| \right) \\ Q(u) &= \sum \left\{ \left| \sigma_{i, \alpha}^l \sigma_{j, \beta}^l \right| : (\sigma_{i, \alpha}^l, \sigma_{j, \beta}^l) \in \mathcal{A}^l \right\} \\ \Upsilon(\mathbf{p}) &= V(u) + \hat{K} \cdot \text{TV}(\Pi) + \check{K} \cdot Q(u), \end{aligned} \quad (4.4)$$

where \mathcal{A}^l denotes the set of approaching waves in the l -th pipe, see [7, Paragraph 7.3], while $\sigma_{i, \alpha}^l$ is the (total) size of the i -wave in the solution of the Riemann problem at x_{α} in the l -th pipe. Note that at $x_{\alpha} = 0$, we consider the Riemann problem at the boundary, according to Section 4.1.1. The constant K_J is defined as in [20, formula (6.2)], \check{K} is as in [20, Paragraph 6] and \hat{K} is as in [2, Section 6].

The lower semicontinuous extension of Υ to all functions with small total variation is achieved in [19] in the case of the Cauchy problem on the whole real line. Here, we use the analogous result on the half line $x > 0$ and the lower semicontinuity of the total variation with respect to the \mathbf{L}^1 norm, see [15] for details.

Moreover, in the proof of the Lipschitz continuous dependence of (4.7) with respect to the initial datum and the condition at the junction, an excellent tool is the stability functional introduced in [8, 37, 38], see also [7, 19]:

$$\Phi(\mathbf{p}, \tilde{\mathbf{p}}) = \sum_{i=1}^2 \sum_{l=1}^n \int_0^{+\infty} \left| q_i^l(x) \right| \mathbf{W}_i^l(x) dx + \bar{K} \left\| \Pi - \tilde{\Pi} \right\|_{\mathbf{L}^1}, \quad (4.5)$$

where u, \tilde{u} are piecewise constant functions in \mathcal{U}_{δ} and we let $(q_1^l(x), q_2^l(x)) = \mathbf{q}(u^l(x), \tilde{u}^l(x))$, see [7, Chapter 8]). The weights \mathbf{W}_i^l are defined by

$$\begin{aligned} \mathbf{W}_1^l(x) &= K \cdot \left(1 + \kappa_1 \mathbf{A}_1^l(x) + \kappa_1 \kappa_2 (\Upsilon(\mathbf{p}) + \Upsilon(\tilde{\mathbf{p}})) \right) \\ \mathbf{W}_2^l(x) &= 1 + \kappa_1 \mathbf{A}_2^l(x) + \kappa_1 \kappa_2 (\Upsilon(\mathbf{p}) + \Upsilon(\tilde{\mathbf{p}})) \end{aligned} \quad (4.6)$$

for suitable positive constants κ_1, κ_2 defined similarly to [7, Chapter 8], see also [19], and K as in [20, Section 6]. Here, Υ is the functional defined in (4.4), while the \mathbf{A}_i^l are defined by

$$\mathbf{A}_i^l(x) = \sum \left\{ \left| \sigma_{k_{\alpha}, \alpha}^l \right| : \begin{array}{l} x_{\alpha} < x, i < k_{\alpha} \leq 2 \\ x_{\alpha} > x, 1 \leq k_{\alpha} < i \end{array} \right\}$$

$$+ \begin{cases} \sum \left\{ \left| \sigma_{i,\alpha}^l \right| : \begin{array}{l} x_\alpha < x, \alpha \in \mathcal{J}(u) \\ x_\alpha > x, \alpha \in \mathcal{J}(w) \end{array} \right\} & \text{if } q_i^l(x) < 0, \\ \sum \left\{ \left| \sigma_{i,\alpha}^l \right| : \begin{array}{l} x_\alpha < x, \alpha \in \mathcal{J}(w) \\ x_\alpha > x, \alpha \in \mathcal{J}(u) \end{array} \right\} & \text{if } q_i^l(x) \geq 0, \end{cases}$$

see [7, Chapter 8]. Note that the lower semicontinuous extension of Φ to all functions with small total variation, defined in [19], keeps all the properties of the original functional Φ . In particular, there exists a constant C such that for all $\mathbf{p}, \tilde{\mathbf{p}} \in \mathcal{D}_\delta$,

$$\frac{1}{C} \left(\|u - \tilde{u}\|_{\mathbf{L}^1} + \|\Pi - \tilde{\Pi}\|_{\mathbf{L}^1} \right) \leq \Phi(\mathbf{p}, \tilde{\mathbf{p}}) \leq C \left(\|u - \tilde{u}\|_{\mathbf{L}^1} + \|\Pi - \tilde{\Pi}\|_{\mathbf{L}^1} \right).$$

Proposition 4.5 *Let $n \in \mathbb{N}$, $n \geq 2$ and f satisfy (\mathbf{F}) at \bar{u} . Let $\bar{\Pi} = \Psi(\bar{u})$. Then, there exist positive δ, L and a semigroup $P: [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$ such that:*

1. $\mathcal{D} \supseteq \text{cl}_{\mathbf{L}^1} \mathcal{D}^\delta$;
2. for all $(u, \Pi) \in \mathcal{D}$, $P_t(u, \Pi) = (S_t(u, \Pi), \mathcal{T}_t \Pi)$, with $P_0 \mathbf{p} = \mathbf{p}$ and for $s, t \geq 0$, $P_s P_t \mathbf{p} = P_{s+t} \mathbf{p}$;
3. for all $(u_o, \Pi) \in \mathcal{D}$, the map $t \mapsto S_t(u_o, \Pi)$ solves

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ \Psi(u(0+, t)) = \Pi(t) \\ u(0, x) = u_o(x) \end{cases} \quad (4.7)$$

according to Definition 2.2;

4. for $\mathbf{p}, \tilde{\mathbf{p}} \in \mathcal{D}$ and $t, \tilde{t} \geq 0$

$$\begin{aligned} \|S_t \mathbf{p} - S_{\tilde{t}} \tilde{\mathbf{p}}\|_{\mathbf{L}^1(\mathbb{R}^+)} &\leq L \cdot \left(\|u - \tilde{u}\|_{\mathbf{L}^1(\mathbb{R}^+)} + \|\Pi - \tilde{\Pi}\|_{\mathbf{L}^1([0, t])} \right) \\ \|S_t \mathbf{p} - S_{\tilde{t}} \mathbf{p}\|_{\mathbf{L}^1(\mathbb{R}^+)} &\leq L \cdot |t - \tilde{t}|. \end{aligned}$$

5. if $\mathbf{p} \in \mathcal{D}$ is piecewise constant then, for $t > 0$ sufficiently small, $S_t \mathbf{p}$ coincides with the juxtaposition of the solutions to Riemann Problems centered at the points of jumps or at the junction;
6. for all $\mathbf{p} \in \mathcal{D}$, the map $t \mapsto \Upsilon(P_t \mathbf{p})$ is non increasing;
7. for all $\mathbf{p}, \tilde{\mathbf{p}} \in \mathcal{D}$, the map $t \mapsto \Phi(P_t \mathbf{p}, P_t \tilde{\mathbf{p}})$ is non increasing;

8. there exist constants $C, \eta > 0$ such that for all $t > 0$, for all $\mathbf{p}, \tilde{\mathbf{p}} \in \mathcal{D}$ and $v \in \mathbf{L}^1(\mathbb{R}^+; \Omega)$ with $\text{TV}(v) < \eta$

$$\begin{aligned} \|S_t \mathbf{p} - S_t \tilde{\mathbf{p}} - v\|_{\mathbf{L}^1(\mathbb{R}^+)} &\leq L \left(\|u - \tilde{u} - v\|_{\mathbf{L}^1(\mathbb{R}^+)} + \left\| \Pi - \tilde{\Pi} \right\|_{\mathbf{L}^1([0, t])} \right) \\ &\quad + C \cdot t \cdot \text{TV}(v). \end{aligned}$$

Thanks to Proposition 4.2, the above result falls within the scope of the theory of initial - boundary value problem for hyperbolic conservation laws, see [1, 2, 3] and [15] for the details.

We now consider the source term.

Proposition 4.6 *Let g satisfy (\mathbf{G}) . For all $(u, \Pi) \in \mathcal{D}$, for all $t_o \in [0, T]$ and all $\tau > 0$ sufficiently small, the following relation holds:*

$$\Upsilon(u + \tau G(t_o, u), \Pi) \leq \Upsilon(u, \Pi) + C \cdot \tau$$

$$\Phi \left((u + \tau G(t_o, u), \Pi), (\tilde{u} + \tau G(t_o, \tilde{u}), \tilde{\Pi}) \right) \leq e^{C\tau} \Phi \left((u, \Pi), (\tilde{u}, \tilde{\Pi}) \right)$$

Proof. We consider piecewise constant functions, leaving the lower semi-continuous extension to general functions to the techniques used in the proof of [17, Lemma 2.3].

For piecewise constant functions, the proof is obtained through a careful control of interactions. Off from the boundary, the same computations of [18, formula (3.6), Lemma 3.6] or [4, Lemma 2.2] hold.

Due to the junction, we have one more term:

$$\begin{aligned} &\Upsilon(u + \tau G(t_o, u), \Pi) - \Upsilon(u, \Pi) \\ &= \mathbf{V}(u + \tau G(t_o, u), \Pi) - \mathbf{V}(u, \Pi) + \tilde{K} \left(\mathbf{Q}(u + \tau G(t_o, u), \Pi) - \mathbf{Q}(u, \Pi) \right) \\ &\leq C\tau + C\tau \left| (G(t_o, u))(0+) \right| \\ &\leq C\tau \end{aligned}$$

The first estimate is thus proved. Concerning the second one, note that the $q_i^l(x)$ are not affected by the presence of the junction, so that exactly the same computations in the case of [17, 18] hold. \square

With the constant C defined above, let

$$\mathcal{D}_t = \left\{ (u, \Pi) \in \mathcal{D} : \Upsilon((u, \Pi)) \leq \delta - C(T - t) \right\}$$

where $T \leq \hat{T}$ and $T \leq \delta/C$.

We now aim at showing that the present situation falls within the scope of [16]. In the metric space X equipped with the \mathbf{L}^1 distance (2.2) introduce the *local flow*

$$F_{t,t_o}(u, \Pi) = \left(S_t(u, \Pi) + tG(t_o, S_t(u, \Pi)), \mathcal{T}_t \Pi \right).$$

In the next proposition, we refer to [16, Definition 2.1 and Condition (D)].

Proposition 4.7 *F is a local flow.*

Proof. For $t_o \in [0, T]$, $\tau \in [0, T - t_o]$ sufficiently small and $u \in \mathcal{D}_{t_o}$, due to the first inequality in Proposition 4.6 we have

$$\begin{aligned} \Upsilon(F_{\tau,t_o}(u, \Pi)) &= \Upsilon\left(S_\tau(u, \Pi) + \tau G(t_o, S_\tau(u, \Pi)), \mathcal{T}_\tau \Pi(t_o)\right) \\ &\leq \Upsilon(P_\tau(u, \Pi)) + C\tau \\ &\leq \Upsilon(u, \Pi) + C\tau \end{aligned}$$

and therefore

$$F_{\tau,t_o}(\mathcal{D}_{t_o}) \subseteq \mathcal{D}_{t_o+\tau}.$$

We now prove the Lipschitz dependence of F from τ and (u, Π) . Note that (G) implies the boundedness of G for $t_o \in [0, T]$ and $u \in \mathcal{D}_{t_o}$. Let $\tau_1, \tau_2 \in [0, T - t_o]$, then, by 4. in Proposition 4.5 and (G)

$$\begin{aligned} d_X(F_{\tau_1,t_o}(u, \Pi), F_{\tau_2,t_o}(u, \Pi)) &\leq \\ &\leq \|S_{\tau_1}(u, \Pi) - S_{\tau_2}(u, \Pi)\|_{\mathbf{L}^1} \\ &\quad + \left\| \tau_1 G(t_o, S_{\tau_1}(u, \Pi)) - \tau_2 G(t_o, S_{\tau_2}(u, \Pi)) \right\|_{\mathbf{L}^1} + \|\mathcal{T}_{\tau_1} \Pi - \mathcal{T}_{\tau_2} \Pi\|_{\mathbf{L}^1} \\ &\leq C \cdot |\tau_2 - \tau_1| + C \text{TV}(\Pi) |\tau_2 - \tau_1| \\ &\leq C \cdot |\tau_2 - \tau_1|. \end{aligned}$$

Similarly, we get for $t_o \in [0, T]$, $\tau \in [0, T - t_o]$ sufficiently small and $(u, \Pi), (\tilde{u}, \tilde{\Pi}) \in \mathcal{D}$

$$\begin{aligned} d_X(F_{\tau,t_o}(u, \Pi), F_{\tau,t_o}(\tilde{u}, \tilde{\Pi})) &\leq \\ &\leq \|S_\tau(u, \Pi) - S_\tau(\tilde{u}, \tilde{\Pi})\|_{\mathbf{L}^1} + \tau \left\| G(t_o, S_\tau(u, \Pi)) - G(t_o, S_\tau(\tilde{u}, \tilde{\Pi})) \right\|_{\mathbf{L}^1} \\ &\quad + \|\mathcal{T}_\tau \Pi - \mathcal{T}_\tau \tilde{\Pi}\|_{\mathbf{L}^1} \\ &\leq C \cdot \|u - \tilde{u}\|_{\mathbf{L}^1} + \left\| \Pi - \tilde{\Pi} \right\|_{\mathbf{L}^1}, \end{aligned}$$

completing the proof. \square

Lemma 4.8 *There exists a constant C such that for all $t_o \in [0, T]$, for all $(u, \Pi), (\tilde{u}, \tilde{\Pi}) \in \mathcal{D}$ and $\varepsilon > 0$ sufficiently small*

$$\Phi \left(F_{\varepsilon, t_o}(u, \Pi), F_{\varepsilon, t_o}(\tilde{u}, \tilde{\Pi}) \right) \leq (1 + C\varepsilon) \Phi \left((u, \Pi), (\tilde{u}, \tilde{\Pi}) \right).$$

This proof follows directly from Proposition 4.6 and 7. in Proposition 4.5.

For $\varepsilon > 0$, recall the definition of the Euler ε -polygonal F^ε generated by F , see [16, Definition 2.2]. Let $k = \lceil \tau/\varepsilon \rceil$, $[\cdot]$ denoting the integer part.

$$F_{\tau, t_o}^\varepsilon(u, \Pi) = F_{\tau - k\varepsilon, t_o + k\varepsilon} \circ \bigcirc_{h=0}^{k-1} F_{\varepsilon, t_o + h\varepsilon}(u, \Pi). \quad (4.8)$$

The conditions that allow to construct a process generated by F that yields solutions to (1.1) are proved to hold in the following proposition.

Proposition 4.9 *The local flow F satisfies the conditions:*

1. *there exists a positive C such that for all $t_o \in [0, T]$, $\tau \in [0, T - t_o]$, $(u, \Pi) \in \mathcal{D}_{t_o}$ and all $k \in \mathbb{N}$ with $(k+1)\tau \in [0, T - t_o]$,*

$$d_X \left(F_{k\tau, t_o + \tau} \circ F_{\tau, t_o}(u, \Pi), F_{(k+1)\tau, t_o}(u, \Pi) \right) \leq C k \tau \tau; \quad (4.9)$$

2. *there exists a positive constant L such that for all $\varepsilon \in [0, \delta]$, for all $t_o \in [0, T]$, $\tau > 0$ sufficiently small and for all $(u, \Pi); (\tilde{u}, \tilde{\Pi}) \in \mathcal{D}_{t_o}$*

$$d_X \left(F_{\tau, t_o}^\varepsilon(u, \Pi), F_{\tau, t_o}^\varepsilon(\tilde{u}, \tilde{\Pi}) \right) \leq L \cdot d_X \left((u, \Pi), (\tilde{u}, \tilde{\Pi}) \right).$$

Proof. Consider the two conditions separately.

1. (We follow here the same steps in the proof of [17, Theorem 1.1]). We refer to the map G introduced in **(G)**. G is \mathbf{L}^1 -bounded, \mathbf{L}^1 -Lipschitz and $\text{TV}(G(u))$ is uniformly bounded for $u \in \mathcal{D}$. Moreover, using 4. and 8. in Proposition 4.5,

$$\begin{aligned} & \left\| F_{k\tau, t_o + \tau} F_{\tau, t_o}(u, \Pi) - F_{(k+1)\tau, t_o}(u, \Pi) \right\|_X = \\ &= \left\| F_{k\tau, t_o + \tau} (S_\tau \mathbf{p} + \tau G(t_o, S_\tau \mathbf{p}), \mathcal{T}_\tau \Pi) \right. \\ & \quad \left. - (S_{(k+1)\tau} \mathbf{p} + (k+1)\tau G(t_o, S_{(k+1)\tau} \mathbf{p}), \mathcal{T}_{(k+1)\tau} \Pi) \right\|_X \\ &= \left\| S_{k\tau} (S_\tau \mathbf{p} + \tau G(t_o, S_\tau \mathbf{p}), \mathcal{T}_\tau \Pi) \right. \\ & \quad \left. + k\tau G(t_o + \tau, S_{k\tau} (S_\tau \mathbf{p} + \tau G(t_o, S_\tau \mathbf{p}), \mathcal{T}_\tau \Pi)) \right. \\ & \quad \left. - (S_{(k+1)\tau} \mathbf{p} + (k+1)\tau G(t_o, S_{(k+1)\tau} \mathbf{p})) \right\|_{\mathbf{L}^1} \\ & \quad + \left\| \mathcal{T}_{k\tau} \mathcal{T}_\tau \Pi - \mathcal{T}_{(k+1)\tau} \Pi \right\|_{\mathbf{L}^1} \end{aligned}$$

$$\begin{aligned}
&\leq \left\| S_{k\tau} (S_\tau \mathbf{p} + \tau G(t_o, S_\tau \mathbf{p}), \mathcal{T}_\tau \Pi) - S_{k\tau} (S_\tau \mathbf{p}, \mathcal{T}_\tau \Pi) - \tau G(t_o, S_{(k+1)\tau} \mathbf{p}) \right\|_{\mathbf{L}^1} \\
&\quad + k\tau \left\| G \left(t_o + \tau, S_{k\tau} (S_\tau \mathbf{p} + \tau G(t_o, S_\tau \mathbf{p}), \mathcal{T}_\tau \Pi) \right) - G(t_o, S_{(k+1)\tau} \mathbf{p}) \right\|_{\mathbf{L}^1} \\
&\leq \left\| \tau G(t_o, S_\tau \mathbf{p}) - \tau G(t_o, S_{(k+1)\tau} \mathbf{p}) \right\|_{\mathbf{L}^1} + C k\tau \text{TV}(\tau G(t_o, S_\tau \mathbf{p})) \\
&\quad + k\tau L_1 \left(\tau + \left\| S_{k\tau} (S_\tau \mathbf{p} + \tau G(t_o, S_\tau \mathbf{p}), \mathcal{T}_\tau \Pi) - S_{k\tau} (S_\tau \mathbf{p}, \mathcal{T}_\tau \Pi) \right\|_{\mathbf{L}^1} \right) \\
&\leq C\tau \left\| S_\tau \mathbf{p} - S_{k\tau} (S_\tau \mathbf{p}, \mathcal{T}_\tau \Pi) \right\|_{\mathbf{L}^1} + C L_2 k\tau \tau \\
&\quad + k\tau L_1 \cdot \left(\tau + C \left\| \tau G(t_o, S_\tau \mathbf{p}) \right\|_{\mathbf{L}^1} \right) \\
&\leq C k\tau \tau.
\end{aligned}$$

2. Let $k = \lceil \tau/\varepsilon \rceil$ with $k \in \mathbb{N}$. By Lemma 4.8 and 7. in Proposition 4.5,

$$\begin{aligned}
&\left\| F_{\tau, t_o}^\varepsilon(u, \Pi) - F_{\tau, t_o}^\varepsilon(\tilde{u}, \tilde{\Pi}) \right\|_X \\
&\leq C \Phi \left(F_{\tau, t_o}^\varepsilon(u, \Pi), F_{\tau, t_o}^\varepsilon(\tilde{u}, \tilde{\Pi}) \right) \\
&\leq C (1 + C(\tau - k\varepsilon)) \Phi \left(F_{k\varepsilon, t_o}^\varepsilon(u, \Pi), F_{k\varepsilon, t_o}^\varepsilon(\tilde{u}, \tilde{\Pi}) \right) \\
&\leq C (1 + C\varepsilon) (1 + C(\tau - k\varepsilon)) \Phi \left(F_{(k-1)\varepsilon, t_o}^\varepsilon(u, \Pi), F_{(k-1)\varepsilon, t_o}^\varepsilon(\tilde{u}, \tilde{\Pi}) \right) \\
&\leq \dots \\
&\leq C (1 + C\varepsilon)^k (1 + C(\tau - k\varepsilon)) \Phi \left((u, \Pi), (\tilde{u}, \tilde{\Pi}) \right) \\
&\leq C e^{C\tau} \Phi \left((u, \Pi), (\tilde{u}, \tilde{\Pi}) \right) \\
&\leq C e^{CT} \left\| (u, \Pi) - (\tilde{u}, \tilde{\Pi}) \right\|_X,
\end{aligned}$$

completing the proof. \square

Proof of Theorem 2.3. The application of [16, Theorem 2.5] yields a process \mathcal{E} , Lipschitz with respect to the \mathbf{L}^1 distance on X , proving 1. to 4. in Theorem 2.3.

The tangency condition 6. follows from [16, (2.9) of Theorem 2.5], indeed

$$\begin{aligned}
&\left\| u(t) - (S_t(u_o, \Pi) + t G(t_o, u_o)) \right\|_{\mathbf{L}^1} \\
&\leq \left\| \mathcal{E}_{t, t_o}(u_o, \Pi) - F_{t, t_o}(u_o, \Pi) \right\|_X \\
&\quad + \left\| F_{t, t_o}(u_o, \Pi) - (S_t(u_o, \Pi) + t G(t_o, u_o), \mathcal{T}_t \Pi) \right\|_X \\
&\leq C t^2 + t \left\| G(t_o, S_t(u_o, \Pi)) - G(t_o, u_o) \right\|_{\mathbf{L}^1} \\
&\leq C t^2.
\end{aligned}$$

Conditions **(W)** and **(Ψ)** are an easy consequence of the tangency condition, see [18, Theorem 1.2].

Finally, by [16, b) in Theorem 2.5], the process \mathcal{E} is Lipschitz, i.e.

$$\left\| \mathcal{E}(\tau, t_o)(u, \Pi) - \mathcal{E}(\tau, t_o)(\tilde{u}, \tilde{\Pi}) \right\|_X \leq L \left(\|u - \tilde{u}\|_{\mathbf{L}^1} + \|\Pi - \tilde{\Pi}\|_{\mathbf{L}^1} \right).$$

The better bound (2.4) follows from the above construction. Indeed, the approximate solution $F_{\tau, t_o}^\varepsilon(u, \Pi)$ depends only on the restriction of Π to $[t_o, t_o + \tau]$. \square

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